# NON-BRANCHING DEGREES IN THE MEDVEDEV LATTICE OF $\Pi^0_1$ CLASSES.

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**Abstract.** A $\Pi^0_1$ class is the set of paths through a computable tree. Given classes $P$ and $Q$, $P$ is Medvedev reducible to $Q$, $P \leq_M Q$, if there is a computably continuous functional mapping $Q$ into $P$. We look at the lattice formed by $\Pi^0_1$ subclasses of $2^\omega$ under this reduction. It is known that the degree of a splitting class of c.e. sets is non-branching. We further characterize non-branching degrees, providing two additional properties which guarantee non-branching: inseparable and hyperinseparable. Our main result is to show that non-branching iff inseparable if hyperinseparable if homogeneous and that all unstated implications do not hold. We also show that inseparable and not hyperinseparable degrees are downward dense.

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**§1. Overview.** A $\Pi^0_1$ class has many equivalent definitions. For the purposes of this paper we will view $\Pi^0_1$ classes as the set of infinite paths through a computable tree. A good introduction and background on $\Pi^0_1$ classes can be found in Cenzer and Jockusch [3]. These classes appear frequently in computable mathematics. A survey of their uses can be found in Cenzer and Remmel [4].

We will be concerned with the Medvedev lattice of non-empty $\Pi^0_1$ subsets of $2^\omega$. In this context we say that a class $P$ is Medvedev below $Q$, written $P \leq_M Q$, if there is a computably continuous functional from $Q$ into $P$, i.e., if it is possible to uniformly compute an element of $P$ given an element of $Q$. A common intuition is to view a class as the solution set to some mathematical problem and $P \leq_M Q$ as implying that solving $Q$ is sufficient to solve $P$. The Medvedev reduction forms a distributive lattice.[1] We will denote this lattice by $\mathcal{L}_M$. This lattice has recently been studied by Simpson [6], Binns [1], and Cenzer and Hinman [2]. In particular, in [1], Binns demonstrated a dense splitting theorem, thus showing

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[1] This lattice is also referred to in the literature as the *strong* lattice of $\Pi^0_1$ classes.

that there are no non-trivial non-splitting degrees. The main purpose of this paper is to investigate the dual, non-branching degrees.

It is an elementary exercise to show that the bottom degree of $\mathcal{L}_M$ does not branch. In [2], Cenzer and Hinman further demonstrated that the *homogeneous* degrees — those of splitting classes of computably enumerable (c.e.) sets — do not branch.

As a convention we say that a degree has some property X if a member has property X. A clopen set $C$ is *good* for a $\Pi^0_1$ class $P$ if $P \cap C$ and $P \cap C^c$ are non-empty, i.e., if $C$ splits $P$ into two proper clopen subclasses.

Call a class $P$ *inseparable* if, for every $C$ good for $P$, $P \cap C \leq_M P \cap C^c$ or $P \cap C^c \leq_M P \cap C$. Thus inseparability states that there is no splitting of $P$ into incomparable clopen subclasses. We will show that inseparability is an invariant of an $\mathcal{L}_M$-degree, and that inseparability is equivalent to being non-branching.

A natural strengthening of inseparability is to change the "or" to an "and". Thus call a class $P$ *hyperinseparable* if, for every $C$ good for $P$, $P \cap C \equiv_M P \cap C^c$. As $P \cap C \geq_M P$ for all $C$, an equivalent statement is $P \cap C \equiv_M P$. Thus hyperinseparability states that all clopen subclasses of $P$ are equivalent to $P$. It is clear that hyperinseparability implies inseparability and thus being non-branching.

A result of Cenzer and Hinman [2] shows that homogeneity implies hyperinseparability. We arrive at the following implications:

$$\text{Homogeneous} \Rightarrow \text{Hyperinseparable} \Rightarrow \text{Inseparable} \Leftrightarrow \text{Non-Branching.}$$

In the latter half of the paper we will show that no additional implications hold, i.e., that these are distinct classes of non-branching degrees. Along the way we will show downward density: below any degree and above $0_M$ there is a degree which is inseparable and not hyperinseparable.

Section 2 will provide definitions, notations, and basic results for the concepts used in this paper. Section 3 will present inseparability and show that it is closed under $\equiv_M$ and equivalent to being non-branching. Section 4 will discuss the stronger condition, hyperinseparability, and show that, while not invariant, it nonetheless strongly influences the members of a degree. Section 5 will review homogeneous classes. Section 6 will show that degrees exist which are inseparable and not hyperinseparable and obey a form of density as mentioned above. Section 7 will show that a degree exists which is hyperinseparable and not homogeneous.

**§2. Definitions and conventions.** The notation used in this paper generally conforms to that found in Cenzer and Hinman [2]. For an overview of the concepts and theory of computability theory see Rogers [5] or Soare [8].

Given a string $\sigma \in 2^{<\omega}$ we denote the length of $\sigma$ by $|\sigma|$. The initial substring relation is denoted by $\prec$. Concatenation of two strings is written $\sigma^\frown\tau$. The empty string is denoted by $\emptyset$, the string of a single 1 by 1, and of a single 0 by 0. Truncation to the first $n$ coordinates is denoted by $\sigma\restriction n$. For $X \in 2^\omega$ we say $\sigma \prec X$ if $\sigma$ is an initial segment of $X$.

A tree, $\mathbb{T}$, is a subset of $2^{<\omega}$ closed under $\prec$. The set of infinite paths through $\mathbb{T}$ is denoted by $[\mathbb{T}]$ and the set of extendible members, those which are initial substrings of members of $[\mathbb{T}]$, is denoted by $\mathrm{Ext}(\mathbb{T})$. For $\sigma \in 2^{<\omega}$ define $\sigma^\frown\mathbb{T} = \{\sigma^\frown\tau : \tau \in \mathbb{T}\}$.

A $\Pi^0_1$ class $P$ is a non-empty subset of $2^\omega$ such that there exists a computable tree $\mathbb{P}$ with $P = [\mathbb{P}]$.[2] We denote the tree of initial substrings of members by $\mathcal{T}_P$. Note that $[\mathcal{T}_P] = P$, $\mathcal{T}_P$ is uniquely determined by $P$, and $\mathcal{T}_P$ is the set of extendible members of any computable tree generating $P$. Similar to the above, define $\sigma^\frown P = [\sigma^\frown\mathcal{T}_P]$.

Define $I(\sigma) = \{X \in 2^\omega : \sigma \prec X\}$. In $2^\omega$, any clopen subset will be a finite union of such cones. A clopen set $C$ will be *good* for a class $P$ if $P \cap C \neq \emptyset \neq P \cap C^c$. By abuse of notation we say that $\sigma \in C$ if it is an initial substring of some $X \in C$. Similarly, if $\mathbb{T}$ is a tree, then $\mathbb{T} \cap C = \{\sigma \in \mathbb{T} : \sigma \in C\}$.

A central concept in this topic is that of a computably continuous functional. Consider the following definition.[3]

Definition 2.1. *A partial computable function $\phi\colon 2^{<\omega} \to 2^{<\omega}$ is a* tree map *if it obeys the following two properties:*

$$\forall\sigma,\tau \in \mathrm{dom}(\phi)\big(\sigma \preceq \tau \Rightarrow \phi(\sigma) \preceq \phi(\tau)\big), \tag{1}$$

$$\forall X \in [\mathrm{dom}(\phi)]\forall n\exists m\big(|\phi(X\restriction m)| > n\big). \tag{2}$$

*A* computably continuous functional *is a function $\Phi\colon 2^\omega \to 2^\omega$ such that there exists a tree map $\phi$ with $\Phi(X) = \bigcup_n \phi(X\restriction n)$.*

The following lemma states that we can assume $\Phi$, and thus $\phi$, to be total. For a proof, see [2].

Lemma 2.2. *Let $P$ and $Q$ be $\Pi^0_1$ classes such that $P \leq_M Q$, then there exists a total computable functional $\Phi : 2^\omega \to 2^\omega$ such that $\Phi(Q) \subseteq P$.*

Thus when we say $\Phi\colon Q \to P$ we mean that there is a total tree map $\phi$ with $\phi(\mathcal{T}_Q) \subseteq \mathcal{T}_P$.

By applying $\leq_M$ to the non-empty $\Pi^0_1$ classes in $2^\omega$ we obtain a degree structure which we denote by $\mathcal{L}_M$. We denote the bottom degree by $\mathbf{0}$ and

[2]This definition is for the purposes of this paper. In full generality a $\Pi^0_1$ class is a (possibly empty) subset of $\omega^\omega$. There are also several equivalent alternatives to the computable tree definition. See [3].

[3]There are alternative, equivalent, definitions. A common one is to define strings as partial computable functions and $\Phi(X)(n) = \phi^X(n)$.

the top degree by **1**. It is useful to observe that **0** is the degree containing exactly all classes with a computable member. The symbols **a**, **b**, and **c** will denote degrees in $\mathcal{L}_M$. For a $\Pi^0_1$ class $P$, the degree of $P$ (under $\leq_M$) will be denoted by $\deg(P)$. Note that $2^\omega$ is a $\Pi^0_1$ class in **0**. For clarity we will denote $2^\omega$ by $0_M$ when using it as a canonical member of **0**.

An immediate but much used lemma is the following.

LEMMA 2.3. *Let $Q$ and $P$ be $\Pi^0_1$ classes with $Q \subseteq P$. Then $Q \geq_M P$.*

PROOF. The identity function serves as a witness. ⊣

The following results from the theory of $\Pi^0_1$ classes will be important.

LEMMA 2.4. *If $\mathbb{P}$ is a co-c.e. tree then there exists a computable tree $\mathbb{Q}$, such that $[\mathbb{P}] = [\mathbb{Q}]$. Furthermore, we can effectively find $\mathbb{Q}$ from $\mathbb{P}$.*

PROOF SKETCH. Let $\{A_s\}_{s\in\omega}$ be an enumeration of $2^{<\omega} \setminus \mathbb{P}$ and $\mathbb{Q} = \{\sigma \colon \forall\tau \preceq \sigma(\tau \notin A_{|\sigma|})\}$. ⊣

DEFINITION 2.5. *Define $P_e$ to be $[\mathbb{T}_e]$ where $\mathbb{T}_e$ is the eth co-c.e. tree.*

**§3. Inseparable degrees.** Before we define inseparability consider the following tree characterization of meets.

DEFINITION 3.1. *Given trees $\mathbb{S}$ and $\mathbb{T}$ define the* tree meet *by*

$$\mathbb{S} \wedge \mathbb{T} = (0^\frown\mathbb{S}) \cup (1^\frown\mathbb{T}).$$

LEMMA 3.2. *If $P \in \boldsymbol{a}$ and $Q \in \boldsymbol{b}$ then $[\mathcal{T}_P \wedge \mathcal{T}_Q] \in \boldsymbol{a} \wedge \boldsymbol{b}$.*

See [2] for a proof and related results. For $\Pi^0_1$ classes $P$ and $Q$ we define $P \wedge Q = [\mathcal{T}_P \wedge \mathcal{T}_Q]$.

With this in hand we see that meets result in incomparable trees connected together into a single tree. Under Medvedev equivalence the actual connection might drift but we can hope it is preserved in some form. Thus to avoid being a proper meet we avoid having incomparable subtrees.

DEFINITION 3.3. *A $\Pi^0_1$ class $P$ is* inseparable *if for all clopen sets $C$ good for $P$, either $P \cap C \leq_M P \cap C^c$ or $P \cap C \geq_M P \cap C^c$.*

To show a degree is non-branching we strive to show that all of its members are inseparable, i.e., do not resemble tree meets. Throughout this paper we will repeatedly be attempting to show properties of every member of a degree. The general technique will be to show that some property of a single member ensures a (possibly weaker) property of every member. In the case of inseparability we have the strongest possible result, namely:

THEOREM 3.4. *Let $Q$ be an inseparable $\Pi^0_1$ class and $P \equiv_M Q$. Then $P$ is inseparable.*

PROOF. Fix $\Phi : P \to Q$ and $\Psi : Q \to P$ as computable functionals witnessing $P \equiv_M Q$. Fix $C$, a clopen set good for $P$ and let $D = \Psi^{-1}(C)$. As $\Psi$ is continuous, $D$ is clopen. Let $i$ be the identity functional. If $Q \cap D = \emptyset$ then $\Psi(Q) \subseteq P \cap C^c$ and $\mathbb{P} \cap C \xrightarrow{i} \mathbb{P} \xrightarrow{\Phi} \mathbb{Q} \xrightarrow{\Psi} \mathbb{P} \cap C^c$ witnesses $P \cap C^c \leq_M P \cap C$. Symmetrically, if $Q \cap D^c = \emptyset$ then $P \cap C \leq_M P \cap C^c$. If $D$ is good for $Q$ then, without loss of generality, $Q \cap D \leq_M Q \cap D^c$. Let $\Omega\colon \mathbb{Q} \cap D^c \to \mathbb{Q} \cap D$ witness this reduction. Note that the predicate $X \in D$ is computable. We can then define the computable functional

$$\Theta(X) = \begin{cases} \Psi\big(\Omega\big(\Phi(X)\big)\big) & \text{if } \Phi(X) \in Q \cap D^c, \\ \Psi\big(\Phi(X)\big) & \text{if } \Phi(X) \in Q \cap D. \end{cases} \tag{3}$$

witnessing $P \cap C \leq_M P \cap C^c$. Thus $P$ is inseparable. ⊣

COROLLARY 3.5. *A degree $\boldsymbol{a} \in \mathcal{L}_M$ is inseparable iff $\boldsymbol{a}$ is non-branching.*

PROOF. We prove the contrapositive for both directions. Assume $\mathbf{a}$ is branching and let $\mathbf{b}, \mathbf{c} \in \mathcal{L}_M$ be such that $\mathbf{b} \perp \mathbf{c}$ and $\mathbf{a} = \mathbf{b} \wedge \mathbf{c}$. Fix $Q \in \mathbf{b}$ and $R \in \mathbf{c}$. Then $Q \wedge R \in \mathbf{a}$ and $I(0)$ witnesses that $Q \wedge R$ is not inseparable. By Theorem 3.4 no member of $\mathbf{a}$ is inseparable, so $\mathbf{a}$ is not inseparable. For the converse assume $\mathbf{a} \in \mathcal{L}_M$ is not inseparable and fix $P \in \mathbf{a}$ and a clopen set $C$ good for $P$ such that $P \cap C \perp_M P \cap C^c$. Let $R = P \cap C$ and $Q = P \cap C^c$ and note that $R$ and $Q$ are $\Pi^0_1$ classes. By Lemma 2.3, $R \geq_M P$ and $Q \geq_M P$. If $R \leq_M P$, say by $\Phi$, then $\Phi \restriction Q$ witnesses $R \leq_M Q$, a contradiction. Thus $R >_M P$ and similarly $Q >_M P$. Let $S = R \wedge Q$ and observe $S \geq_M P$. Define the computable functional

$$\Psi(X) = \begin{cases} 0^\frown X & \text{if } X \in C, \\ 1^\frown X & \text{if } X \in C^c. \end{cases} \tag{4}$$

Observe $\Psi$ witnesses $S \leq_M P$ and thus $S \equiv_M P$. It follows that $\mathbf{a} = \deg(R) \wedge \deg(Q)$ and $\deg(R) \perp \deg(Q)$. Thus $\mathbf{a}$ is branching. ⊣

**§4. Hyperinseparable degrees.** We strengthen inseparability by requiring reductions in both directions.

DEFINITION 4.1. *A $\Pi^0_1$ class $P$ is* hyperinseparable *if for any clopen set $C$ good for $P$, $P \cap C \equiv_M P \cap C^c$.*

Observe that replacing $P \cap C \equiv_M P \cap C^c$ with $P \cap C \equiv_M P$ results in an equivalent definition. Hyperinseparability claims that any clopen subclass "looks" the same as the whole class.

We might hope that, like inseparability, hyperinseparability is an invariant of a degree. This is not the case.

LEMMA 4.2. *Let $\boldsymbol{a} \in \mathcal{L}_M$. If $\boldsymbol{a} < \boldsymbol{1}$ then $\boldsymbol{a}$ has a non-hyperinseparable member.*

Proof. Fix $P \in \mathbf{a}$ and $Q >_M P$. Then $P \wedge Q \in \mathbf{a}$ and $I(0)$ witnesses that $P \wedge Q$ is not hyperinseparable as $(P \wedge Q) \cap I(0) \equiv_M P <_M Q \equiv_M (P \wedge Q) \cap I(1)$. ⊣

There is, however, a weaker property of all members of a hyperinseparable degree.

Theorem 4.3. *Let $Q$ be a hyperinseparable $\Pi^0_1$ class and $P \equiv_M Q$. Then there exists a $\Pi^0_1$ class $R \subseteq P$ such that $R \equiv_M P$ and $R$ is hyperinseparable.*

Proof. Let $\Phi\colon P \to Q$ and $\Psi\colon Q \to P$ witness $P \equiv_M Q$. Let $R = \Psi(Q)$ and note that $R$ is a $\Pi^0_1$ subclass of $P$. By Lemma 2.3, $R \geq_M P$, and the map $\Psi(\Phi(\cdot))$ witnesses $R \leq_M P$, thus $R \equiv_M P$. Fix a clopen set $C$ good for $R$ and let $D = \Psi^{-1}(C)$. As $\Psi$ is onto $R$, $D$ is good for $Q$. By hyperinseparability there exists $\Omega\colon Q \cap D \to Q \cap D^c$. Define

$$\Theta(X) = \begin{cases} \Psi\big(\Phi(X)\big) & \text{if } \Phi(X) \in Q \cap D^c, \\ \Psi\big(\Omega(\Phi(X))\big) & \text{if } \Phi(X) \in Q \cap D. \end{cases} \tag{5}$$

Thus $R \cap C^c \leq_M R \cap C$. A symmetric argument replacing $\Omega$ with $\Omega'\colon Q \cap D^c \to Q \cap D$ shows $R \cap C^c \geq_M R \cap C$. As $C$ was arbitrary, $R$ is hyperinseparable. ⊣

This result is strengthened by the following lemma which states that such a class, one with an equivalent hyperinseparable subclass, essentially behaves as a hyperinseparable class.

Lemma 4.4. *Let $P$ be a $\Pi^0_1$ class and $R \subseteq P$ with $R \equiv_M P$. If $C$ is a clopen set which is good for both $P$ and $R$ such that $R \cap C \equiv_M R \cap C^c$ then $P \cap C \equiv_M P \cap C^c$.*

Proof. Let $\Omega\colon P \to R$ witness $R \leq_M P$. Let $\Phi\colon R \cap C \to R \cap C^c$ and $\Psi\colon R \cap C^c \to R \cap C$ witness $R \cap C \equiv_M R \cap C^c$. Define

$$\Theta(X) = \begin{cases} \Phi\big(\Omega(X)\big) & \text{if } \Omega(X) \in R \cap C, \\ \Omega(X) & \text{if } \Omega(X) \in R \cap C^c. \end{cases} \tag{6}$$

Observe that $\Theta$ witnesses $P \cap C^c \leq_M P \cap C$. A symmetric argument with $\Psi$ in place of $\Phi$ demonstrates that $P \cap C \leq_M P \cap C^c$. Thus $P \cap C \equiv_M P \cap C^c$. ⊣

A way to think of this Lemma is as follows. If $P$ is a member of a hyperinseparable degree then it has some hyperinseparable core $R$. Any clopen set good for $P$ which does not ignore $R$ (either by avoiding it or absorbing it) splits $P$ into two equivalent pieces.

**§5. Homogeneous degrees.** So far we have not proved the existence of any non-trivial non-branching degrees. It is an easy exercise to show that the bottom degree, **0**, is non-branching. The top degree, **1**, is trivially non-branching. Both **0** and **1** are examples of homogeneous classes. In [2] Cenzer and Hinman show that all homogeneous degrees are non-branching.

DEFINITION 5.1. [2, Def. 8] *A tree* $\mathbb{P}$ *is* homogeneous *if*

$$\forall \sigma, \tau \in \mathbb{P}\ \forall i \in 2\big[|\sigma| = |\tau| \Rightarrow (\sigma ^\frown i \Leftrightarrow \tau ^\frown i)\big].$$

*A class $P$ is* homogeneous *if $\mathcal{T}_P$ is.*

This definition is very closely tied to separating classes. If $A$ and $B$ are subsets of $\omega$ then we define the *separating class* $\mathcal{S}(A, B) = \{C : A \subseteq C \subseteq \omega \setminus B\}$. In the case that $A$ and $B$ are c.e., $\mathcal{S}(A, B)$ is a $\Pi^0_1$ class [2].

LEMMA 5.2. [2, Prop. 9] *For any $\Pi^0_1$ class $P$,*

$$P \text{ is homogeneous} \Leftrightarrow P \text{ is a c.e. separating class.}$$

Note that **0** is the degree of the separating class of $\emptyset$ and $\omega$ and **1** is the degree of the separating class of $\{n\colon \{n\}(n)\downarrow= 0\}$ and $\{n\colon \{n\}(n)\downarrow= 1\}$ [7].

Homogeneous classes have the property of being closed under string splice operations.

DEFINITION 5.3. *Given $\sigma, \tau \in 2^{<\omega}$ define $\tau$* splice *$\sigma$, denoted by $\tau \,/\, \sigma$, as*

$$(\tau \,/\, \sigma)(i) = \begin{cases} \sigma(i) & \text{if } i < |\sigma|, \\ \tau(i) & \text{if } |\sigma| \le i < |\tau|, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

LEMMA 5.4. *For $\sigma \in 2^{<\omega}$ the function $\tau \mapsto \tau \,/\, \sigma$ is a computable tree map.*

PROOF. Let $\phi(\tau) = \tau \,/\, \sigma$. That $\phi$ is computable is immediate. Fix $\tau, \tau' \in 2^{<\omega}$ with $\tau \preceq \tau'$. By definition $\phi(\tau) \restriction |\sigma| = \phi(\tau') \restriction |\sigma|$. For $|\sigma| \le i < |\tau'|$, $\phi(\tau')(i) = \tau'(i) = \tau(i) = \phi(\tau)(i)$. Thus $\phi(\tau) \preceq \phi(\tau')$. Now fix $X \in 2^\omega$ and $n$. Let $m > n$, then $|\phi(X \restriction m)| = |X \restriction m| = m > n$. ⊣

LEMMA 5.5. *If $P$ is a homogeneous $\Pi^0_1$ class, then for all $\sigma, \tau \in \mathcal{T}_P$, $\tau \,/\, \sigma \in \mathcal{T}_P$.*

PROOF. Fix $\sigma$ and induct on $|\tau|$. By assumption $\sigma \in \mathcal{T}_P$, thus for $|\tau| \le |\sigma|$ the conclusion holds. Now fix $\tau$ with $n = |\tau| > |\sigma|$ and assume the result holds for $\tau'$ with $|\tau'| < n$. Then $\tau \restriction (n-1) \,/\, \sigma \in \mathcal{T}_P$. If $\tau = \tau \restriction (n-1) ^\frown 0$ then, by homogeneity, $(\tau \restriction (n-1) \,/\, \sigma) ^\frown 0$ is in $\mathcal{T}_P$ but this is just $\tau \,/\, \sigma$. Similarly if $\tau = \tau \restriction (n-1) ^\frown 1$, then $\tau \,/\, \sigma = (\tau \restriction (n-1) \,/\, \sigma) ^\frown 1$ is in $\mathcal{T}_P$. ⊣

Homogeneous classes fit into our context by the following lemma.

LEMMA 5.6. [2, Lemma 6, rephrased] *If $P$ is a homogeneous $\Pi^0_1$ class then $P$ is hyperinseparable.*

The following proof is essentially equivalent to that of [2] but phrased in terms of Lemma 5.5.

PROOF. Fix a clopen set $C$ good for $P$. Fix $\sigma \in \mathcal{T}_P$ such that $I(\sigma) \subseteq C$. Then, by Lemma 5.5, $\tau \mapsto \tau / \sigma$ witnesses $P \cap I(\sigma) \leq_M P$ and thus $P \cap C \leq_M P$. By Lemma 2.3 $P \cap C \geq_M P$, thus $P \cap C \equiv_M P$ and $P$ is hyperinseparable. ⊣

**§6. An inseparable and not hyperinseparable degree.** In this section we will demonstrate the existence of degrees which are inseparable (and thus non-branching) and not hyperinseparable. We need to construct a degree which has an inseparable member and every member is not hyperinseparable. This construction can be done with a single $\Pi^0_1$ class.

LEMMA 6.1. *Let $P$ be a $\Pi^0_1$ class. If for any clopen set $C$ good for $P$, $P \cap C <_M P \cap C^c$ or $P \cap C >_M P \cap C^c$, then* $\deg(P)$ *is inseparable and not hyperinseparable.*

PROOF. That $\deg(P)$ is inseparable is immediate. Assume $\deg(P)$ is hyperinseparable. By Theorem 4.3 there exists $Q \subseteq P$ such that $Q \equiv_M P$ and $Q$ is hyperinseparable. Fix a clopen set $C$ good for $Q$ and observe that $C$ is also good for $P$. As $Q$ is hyperinseparable, $Q \cap C \equiv_M Q \cap C^c$ and thus, by Lemma 4.4, $P \cap C \equiv_M P \cap C^c$, a contradiction. Thus $\deg(P)$ cannot be hyperinseparable. ⊣

Our goal is to construct a class for which every clopen splitting reduces in exactly one direction. In essence we will embed a uniform descending chain into the class and show that if a clopen subclass contains the tail then it reduces from but not to the remainder of the class.

Using a density theorem such as that of Binns [1] and paying close attention to effectiveness we arrive at the following result.

LEMMA 6.2. *Given indices for $\Pi^0_1$ classes $Q$ and $P$ with $Q >_M P$ and an index for $\Phi : Q \to P$ witnessing $Q \geq_M P$, we can effectively find an index for a total computable function $f$ such that*

$$Q >_M P_{f(0)} >_M P_{f(1)} > \ldots >_M P. \tag{7}$$

PROOF. Observe that the proof in [1] is effective. ⊣

We now turn to the construction of our class.

THEOREM 6.3. *Given $\boldsymbol{b} \in \mathcal{L}_M$ there exists $\boldsymbol{a} \in \mathcal{L}_M$ such that $\boldsymbol{0} <_M \boldsymbol{a} <_M \boldsymbol{b}$ and $\boldsymbol{a}$ is inseparable and not hyperinseparable.*

PROOF. Fix $Q \in \mathbf{b}$. By Lemma 6.2 there exists a computable function $f$such that $Q >_M P_{f(0)} >_M P_{f(1)} >_M \ldots >_M 0_M$. Let $\mathbb{P}_{f(i)}$ be a tree such that $P_{f(i)} = [\mathbb{P}_{f(i)}]$ and similarly for $\mathbb{Q}$. Fix $X$, a c.e. set $>_T \emptyset$. We will build a tree $\mathbb{R}$, a set $Y$, and strings $\{\beta_i\}_{i\in\omega}$,$\{\delta_i\}_{i\in\omega}$ with the following structure:

$$\lim_i \beta_i = Y, \tag{8a}$$

$$\delta_i \succeq \beta_{i-1} \text{ for } i > 0, \tag{8b}$$

$$X \equiv_T Y, \tag{8c}$$

$$R = [\mathbb{R}] = \{Y\} \cup \Big(\bigcup_{i\in\omega} \beta_i \frown P_{f(i)}\Big) \cup \Big(\bigcup_{i\in\omega} \delta_i \frown P_{f(i)}\Big). \tag{8d}$$

Furthermore, (8d) will be a disjoint union. We will show that $\mathbf{a} = \deg(R)$ satisfies our theorem.

For each $i$ we have a strategy with two possible states, *wait* and *stop*. Our construction proceeds in stages. At each stage some finite number of strategies will be active. Strategies start in state *wait* and may move to state *stop* at some later stage. Once in state *stop* a strategy will remain in that state unless injured. When a strategy $i$ moves from state *wait* to state *stop* it will injure all strategies $j$ with $j > i$. Our argument thus proceeds in typical finite injury fashion; for any given strategy there will be a finite stage after which the strategy is never again injured.

Let $\mathbb{R}_s$, $\beta_{i,s}$, and $\delta_{i,s}$ denote $\mathbb{R}$, $\beta_i$, and $\delta_i$ at the end of stage $s$, respectively. To ensure that $\mathbb{R}$ is computable we construct an increasing total computable function, $l(s)$, such that $\sigma \in \mathbb{R}_s \setminus \mathbb{R}_{s-1}$ iff $l(s-1) < |\sigma| \leq l(s)$. We will define $m(s)$ such that strategy $i$ is active at stage $s$ iff $i \leq m(s)$. Fix an enumeration $\{X_s\}_{s\in\omega}$ of $X$ such that $|X_s \setminus X_{s-1}| = 1$.

Each strategy $i$ will have two strings $\beta_i$ and $\delta_i$ and above each it will build a copy of $\mathbb{P}_{f(i)}$. So long as $i \notin X$ the construction for strategies $j > i$ continues above $\beta_i$. If $i$ enters $X$ then the work above $\beta_i$ will be abandoned, $\beta_i$ and $\delta_i$ will swap roles, and construction continued above the new $\beta_i$. In this manner we encode $X$ in $Y = \lim_i \beta_i$. Strategy $i$ is in state *wait* while it is waiting for $i$ to enter $X$. Once $i$ enters $X$ it will change to state *stop*.

To begin our construction let $l(-1) = 0$, $m(-1) = -1$, and $\mathbb{R}_{-1} = \emptyset$.

Assume we have run our construction to stage $s$. Thus $\beta_{i,t}$, $\delta_{i,t}$, $l(t)$, $m(t)$, and $\mathbb{R}_t$ are defined for all $t < s$. Let $x \in X_s \setminus X_{s-1}$ (recall that such exists and is unique). If $x \leq m(s-1)$ then strategy $x$ needs to change state and injure all higher strategies. In such a case let $\beta_{i,s} = \delta_{i,s-1}$, $\delta_{i,s} = \beta_{i,s-1}$, and $j = x + 1$. Here $j$ denotes next strategy to activate. If $x > m(s-1)$ then simply let $j = m(s-1) + 1$.

Having dealt with any possible injury we are ready to expand our tree. We want to activate strategy $j$ and then let all active strategies grow. Fix

$\sigma$ and $\tau$ minimal such that $l(s-1) < |\beta_{j-1,s-1}|+|\sigma| < |\beta_{j-1,s-1}|+|\tau|$ and $\sigma$ and $\tau$ are leaves of $\beta_{j-1,s-1}{}^\frown\mathbb{P}_{f(j)}$. Such $\sigma$,$\tau$ exist as $[\mathbb{P}_{f(j)}] >_M 0_M$ and, furthermore, can be found computably from $j$ and $\beta_{j-1,s-1}$. If $j \notin X$ then let $\beta_{j,s} = \sigma$ and $\delta_{j,s} = \tau$. If $j \in X$ then let $\beta_{j,s} = \tau$ and $\delta_{j,s} = \sigma$. Let $m(s) = j$ and $l(s) = \max\{|\delta_{j,s}|, |\beta_{j,s}|\}$. For all $i < j$, let $\beta_{i,s} = \beta_{i,s-1}$, $\delta_{i,s} = \delta_{i,s-1}$. Let

$$\mathbb{R}_s = \mathbb{R}_{s-1} \cup$$
$$\{\sigma\colon \sigma \preceq \beta_{j,s} \text{ and } l(s-1) < |\sigma|\} \cup \tag{9a}$$
$$\{\sigma\colon \sigma \preceq \delta_{j,s} \text{ and } l(s-1) < |\sigma|\} \cup \tag{9b}$$
$$\{\beta_{i,s}{}^\frown\sigma\colon i \leq j,\ \sigma \in \mathbb{P}_{f(i)},\ \text{and } l(s-1) < |\sigma| + |\beta_{i,s}| \leq l(s)\} \cup \tag{9c}$$
$$\{\delta_{i,s}{}^\frown\sigma\colon i \leq j,\ \sigma \in \mathbb{P}_{f(i)},\ \text{and } l(s-1) < |\sigma| + |\delta_{i,s}| \leq l(s)\}. \tag{9d}$$

Claim. *For each $i$ there exists $s$ such that strategy $i$ is not injured after stage $s$.*

Proof. Each strategy $i$ only injures strategies $j$ with $j > i$. Thus strategy 0 is never injured. Assume the claim holds for all strategies $h < i$ and fix $r$ such that no strategy $h < i$ is injured after stage $r$. If any strategy $g < h$ changed state it would injure strategy $h$, thus no strategy $g < h$ can change state after stage $r$. Strategy $h$ will change state at most once after stage $r$. Fix stage $s > r$ such that $h$ does not change state after stage $s$. So no strategy $h < i$ will change state after stage $s$ and thus strategy $i$ will not be injured after stage $s$. The results follows by induction. ⊣

For any strategy $i$ we can fix $s$ as above and fix $s' \geq s$ such that $X_{s'}(i) = X(i)$. Observe that strategy $i$ will not be injured or change state after stage $s'$.

Claim. $\lim_s m(s) = \infty$ *and* $\lim_s l(s) = \infty$.

Proof. Observe that $l(s)$ is strictly increasing. Thus $\lim_s l(s) = \infty$. The function $m(s)$ only decreases when a strategy $i$ changes state at which point it decreases to $i+1$. If no strategy changes state then $m(s)$ increases by 1. For any $n$, fix $s$ such that strategy $n$ is not injured after stage $s$. Then no strategy $j < n$ will need to change state after stage $s$. If $m(s) < n$ then no strategy changes state and $m(s+1) = m(s)+1$. This continues until a stage $t$ with $m(t) > n$. No strategy below $n$ will need to change state at a later stage so for all $r > t$, $m(r) \geq n$. ⊣

As a corollary, $\mathbb{R}$ is infinite and thus $R = [\mathbb{R}]$ is a non-empty $\Pi^0_1$ class.

Claim. *For all $i$, $\beta_i = \lim_s \beta_{i,s}$ and $\delta_i = \lim_s \delta_{i,s}$ exist.*

Proof. Fix $s$ such that strategy $i$ is not injured and does not change state after stage $s$. Then $\beta_{i,t} = \beta_{i,s}$ for all $t > s$ and similarly for $\delta_{i,t}$. ⊣

CLAIM. *$\beta_{i-1} \prec \beta_i$ and $\beta_{i-1} \prec \delta_i$.*

PROOF. Fix $i$ and observe that $\beta_{i-1}$ will only change value when strategy $i-1$ is injured or changes state. In both cases strategy $i$ will be injured and any value of $\beta_i$ destroyed. Thus $\beta_i$ will settle down only after $\beta_{i-1}$ does. As $\beta_i$ is chosen to extend $\beta_{i-1}$ the first result holds. A symmetric argument shows the second result. $\dashv$

As a corollary, $Y = \lim_i \beta_i$ exists.

CLAIM. *$X \equiv_T Y$.*

PROOF. Assume we have an oracle $X$ and wish to compute whether $x \in Y$. Run the computation until a stage $s$ such that there exists $i$ with $\beta_{i,s}$ defined, $|\beta_{i,s}| > x$, and $X \upharpoonright i = X$. We can compute such a stage knowing $X$. As no $h \leq i$ will enter $X$, strategy $i$ will not be injured or change state after stage $s$. Thus $Y(x) = \beta_{i,s}(x)$ and $Y \leq_T X$. For the converse assume we have an oracle for $Y$ and wish to compute $x \in X$. Run the computation until a stage $s$ such that $\beta_{x,s} \prec Y$. Then $X(x) = X_s(x)$ and $X \leq_T Y$. So $X \equiv_T Y$. $\dashv$

CLAIM. *$R = \{Y\} \cup \left(\bigcup_{i\in\omega} \beta_i \frown P_i\right) \cup \left(\bigcup_{i\in\omega} \delta_i \frown P_i\right)$ and this union is disjoint.*

PROOF. As notation define $\mathbb{T}\upharpoonright n = \{\sigma \in \mathbb{T} : |\sigma| \leq n\}$. Define

$$\mathbb{T}_s = \{\sigma : \sigma \preceq \beta_{m(s),s}\} \cup \bigcup_{i\leq m(s)} \beta_{i,s} \frown \mathbb{P}_{f(i)}\upharpoonright l(s) \cup \bigcup_{i\leq m(s)} \delta_{i,s} \frown \mathbb{P}_{f(i)}\upharpoonright l(s). \tag{10}$$

We will show that for all $s$, $\mathbb{T}_s \subseteq \mathbb{R}_s$, and that for all $\sigma \in \mathbb{R}_{s+1} \setminus \mathbb{R}_s$, $\sigma$ extends some path in $\mathbb{T}_s$. Thus $\mathrm{Ext}(\mathbb{R}) = \lim_s \mathbb{T}_s$ and the claim follows.

Equation (10) is trivially true for $\mathbb{T}_{-1}$. Assume (10) holds for all $\mathbb{T}_t$ with $t < s$. If there is a strategy $i$, $i \leq m(s-1)$, which needs to change state at stage $s$, then we drop $m(s)$ down to $i+1$, and swap $\beta_i$ and $\delta_i$. As both $\beta_{i,s-1}$ and $\delta_{i,s-1}$ are in $\mathbb{T}_{s-1}$ so are $\beta_{i,s}$ and $\delta_{i,s}$. If no strategy needs to change states then $m(s) = m(s-1)+1$. In either case, $m(s)$ is one more than the largest active (uninjured) strategy. Strategy $m(s)$ chooses $\sigma$ and $\tau$ which are leaves of $\mathbb{R}_{m(s)}$ and are longer than $l(s-1)$. By assumption all initial substrings of $\sigma$ and $\tau$ up to length $l(s-1)$ are in $\mathbb{T}_{s-1}$ and by (9a) and (9b) we will have all initial substrings of $\sigma$ and $\tau$ in $\mathbb{R}_s$. The strings $\beta_{m(s),s}$ and $\delta_{m(s),s}$ are chosen from $\sigma$ and $\tau$. Finally, (9c) and (9d) fill in our trees up to length $l(s)$. Strategies only add strings longer than $l(s-1)$ at stage $s$. Thus anytime a string in $\mathbb{T}_{s-1}$ is not extended at stage $s$ it will never be extended and thus is not in $\mathrm{Ext}(\mathbb{R})$. Conversely, only strings in $\mathbb{T}_{s-1}$ are extended at stage $s$, so any member of $\mathrm{Ext}(\mathbb{R})$ must be in every $\mathbb{T}_s$.

To see that the union is disjoint observe that $\beta_i$ and $\delta_i$ are chosen from the leaves of $\mathbb{P}_{f(i-1)}$. Thus $\mathbb{P}_{f(i-1)}$ will not provide any ancestors of $\beta_i$ or $\delta_i$. The result follows by induction. ⊣

We now prove that $R$ and $\mathbf{a}$ have the desired properties.

Recall that $Q \subseteq R \Rightarrow Q \geq_M R$ and $\sigma^\frown Q \equiv_M Q$.

Claim. *For any clopen set $C$ good for $R$, $R \cap C <_M R \cap C^c$ or $R \cap C >_M R \cap C^c$.*

Proof. Fix a clopen set $C$ good for $R$. Without loss of generality we can assume $Y \in P \cap C^c$. Let $G = \bigcup_{i \in \omega}\{\beta_i, \delta_i\}$. Let $A = \{i : (\gamma^\frown P_{f(i)}) \cap (R \cap C) \neq \emptyset$ for some $\gamma \in G\}$ and $B = \{\gamma \in G : \exists i(\gamma^\frown P_{f(i)}) \cap (R \cap C) \neq \emptyset\}$. As $Y \in R \cap C^c$, both $A$ and $B$ are finite and co-infinite. In addition, every string in $\mathbb{R} \cap C$ is either a prefix of some $\gamma \in B$ or of the form $\gamma^\frown \varepsilon$ for some $\gamma \in B$, $\varepsilon \in \mathbb{P}_{f(i)}$, and $i \in A$. Fix $n > \sup A$, and, for all $i \in A$, let $\Theta_i$ be a computably continuous functional with $\Theta_i \colon P_{f(i)} \to P_{f(n)}$. Fix $\sigma \in G$ such that $\sigma^\frown P_{f(n)} \subseteq R$.

We can now define $\Phi\colon R \cap C \to R \cap C^c$ as follows. Fix $U \in R \cap C$. Let $\gamma \in G$, $i$, and $V \in P_{f(i)}$ be such that $U = \gamma^\frown V$. Let $\Phi(U) = \sigma^\frown \Theta_i(V)$. Observe that $\Phi$ only used $A$, $B$, $\Theta_i$ for $i \in A$, and $\sigma$. This is a finite amount of information, so $\Phi$ is a computable functional. Thus $R \cap C \geq_M R \cap C^c$.

Let $m = \sup A$ and observe that $R \cap C \equiv_M \bigwedge_{i \in A}(P_{f(i)} \cap C) \geq_M \bigwedge_{i \in A} P_{f(i)} \equiv_M P_{f(m)}$. If $R \cap C^c \geq_M R \cap C$, then $P_{f(n)} \geq_M R \cap C^c \geq_M R \cap C \equiv_M P_{f(m)}$. But $n > m$ so $P_{f(n)} <_M P_{f(m)}$, a contradiction. Thus $R \cap C^c \not\geq_M R \cap C$. So $R \cap C >_M R \cap C^c$. ⊣

By Lemma 6.1, $\mathbf{a}$ is inseparable and not hyperinseparable.

Claim. $\boldsymbol{0} < \boldsymbol{a} < \boldsymbol{b}$.

Proof. That $\mathbf{0} < \mathbf{a}$ is immediate as 0 is homogeneous and thus hyperinseparable and $\mathbf{a}$ is not. Let $C = I(\delta_0)$ and note $Y \in R \cap C^c$ so $R \leq_M R \cap C^c <_M P \cap C \equiv_M P_{f(0)} <_M Q$ and $\mathbf{a} < \mathbf{b}$. ⊣

This concludes the proof of the theorem. ⊣

We remark in passing that by bounding our sequence $\geq_M P_\omega$ for some $P_\omega$ it is possible to ensure that $\mathbf{a} \geq \deg_w(P)$. where $\deg_w$ is the Muchnik (also called weak) degree.

**§7. A hyperinseparable and not homogeneous degree.** In this section we will construct a degree which is hyperinseparable and not homogeneous. We first explore a condition which guarantees hyperinseparability and is easy to work into a construction. With that in hand we explore certain properties of homogeneous degrees and find a specific, if technical, property which can be diagonalized against. We then build such a class via a finite injury construction.

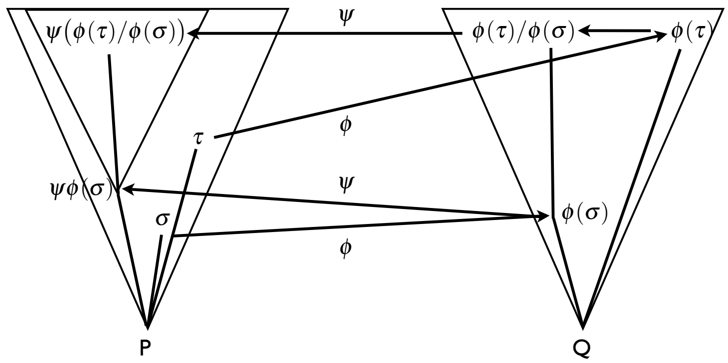


FIGURE 1. Effectiveness in homogeneous degrees

Consider the following generalization of homogeneity.

DEFINITION 7.1. *For a tree* $\mathbb{T}$, $n$ *is a* duplication level *if*

$$\forall\sigma,\tau\in\mathbb{T}\big[(|\sigma|=n \text{ and } |\tau|\geq n)\Rightarrow(\tau\,/\,\sigma\in\mathbb{T})\big].$$

Homogeneity is equivalent to every level being a duplication level.

LEMMA 7.2. *If $P$ is a $\Pi^0_1$ class and $\mathcal{T}_P$ has an infinite number of duplication levels then $P$ is hyperinseparable.*

PROOF. Fix a clopen set $C$ good for $P$ and fix $m$, $n$, and $\alpha_i$ such that $C=\bigcup_{i<m}I(\alpha_i)$ with $|\alpha_i|=n$. Let $n'\geq n$ be a duplication level of $P$ and choose a $\sigma$ such that $\sigma\succeq\alpha_i$ for some $i$ and $|\sigma|=n'$. Then $\tau\mapsto\tau\,/\,\sigma$ witnesses $P\cap I(\sigma)\leq_M P\cap C$ and thus $P\cap C^c\leq_M P\cap C$. A symmetric argument shows $P\cap C\leq_M P\cap C^c$. ⊣

We now turn to properties of homogeneous degrees relating to effectiveness. Consider the following example, illustrated in Figure 1. Fix a homogeneous class $Q$ and $P\equiv_M Q$ with $\Phi\colon P\to Q$ and $\Psi\colon Q\to P$. For any $\sigma\in\mathcal{T}_P$ we define

$$\theta_\sigma(\tau)=\psi\big(\phi(\tau)\,/\,\phi(\sigma)\big).$$

As $\phi$ and $\psi$ are computable and by Lemma 5.4, $\theta_\sigma$ is a computable tree map. Note that an index for $\theta_\sigma$ can be computed from $\sigma$. For $C=I(\psi(\phi(\sigma)))$ we have $\Theta_\sigma\colon P\cap C^c\to P\cap C$. In this manner, in homogeneous degrees, we can effectively convert members of $\mathcal{T}_P$ into witnesses of inseparability. In this simple form we have very little control over what clopen subclasses we are reducing between but it serves as an initial example of the technique.

The purpose behind the following definitions and lemmas is to do something similar to the above except that we constrain $\sigma$ and $C$. Specifically, for a given level $n$, we want $C \subseteq I(\sigma \upharpoonright n)$, i.e., $C$ and $\sigma$ in a single cone generated at level $n$.

Rather than diagonalize against possible classes $Q$, we will diagonalize against pairs of functions witnessing the equivalence. In all that follows, $\phi$ and $\psi$ should be thought of as possible witnesses $\Phi\colon P \to Q$ and $\Psi\colon Q \to P$.

Definition 7.3. *For a $\Pi^0_1$ class $P$, and functions $\phi, \psi\colon 2^{<\omega} \to 2^{<\omega}$ define*

$$\begin{aligned} f_{\phi,\psi}(n) &= \mu\sigma[\sigma \in \mathcal{T}_P \text{ and } |\sigma| > n \text{ and} && (11)\\ &\qquad \exists i(|\sigma| \geq i > 0 \text{ and } |(\psi\phi)^i(\sigma)| > n \text{ and } (\psi\phi)^i(\sigma)\upharpoonright n = \sigma\upharpoonright n)], \\ \operatorname{ind}_{\phi,\psi}(n) &= \mu i[(\psi\phi)^i(f_{\phi,\psi}(n))\upharpoonright n = f_{\phi,\psi}(n)\upharpoonright n], && (12)\\ \hat{f}_{\phi,\psi}(n) &= (\psi\phi)^{\operatorname{ind}_{\phi,\psi}(n)}(f_{\phi,\psi}(n)), && (13)\\ \theta_{\phi,\psi}(n,\sigma,\tau) &= \psi\big(\phi(\tau)\,/\,\phi((\psi\phi)^{\operatorname{ind}_{\phi,\psi}(n)-1}(\sigma))\big). && (14)\end{aligned}$$

*We write $f_{\phi,\psi}(n)\downarrow$ when a valid $\sigma$ exists and $f_{\phi,\psi}(n)\uparrow$ otherwise, and similarly for the other functions.*

In our context, $\sigma = f_{\phi,\psi}(n)$ is the element of $\mathcal{T}_P$ which we use to generate a function, $\theta_{\phi,\psi}(n,\sigma,\cdot)$, mapping into $I(\hat{f}_{\phi,\psi}(n))$. Note that when $\phi$ and $\psi$ are total computable functions $f_{\phi,\psi}$, $\operatorname{ind}_{\phi,\psi}$, and $\hat{f}_{\phi,\psi}$ are all computable, and $\Theta_{\phi,\psi}$ is computably continuous.

Lemma 7.4. *If $P$ and $Q$ are $\Pi^0_1$ classes with $\Phi\colon P \to Q$ and $\Psi\colon Q \to P$ computable functionals, then $f_{\phi,\psi}(n)\downarrow$ for all $n$. Furthermore, if $Q$ is homogeneous, then*

$$\begin{aligned} \forall n\big[&\theta_{\phi,\psi}(n, f_{\phi,\psi}(n), \emptyset)\upharpoonright n = f_{\phi,\psi}(n)\upharpoonright n \text{ and} \\ &\Theta_{\phi,\psi}(n, f_{\phi,\psi}(n), \cdot)\colon P \to P\big]. \end{aligned} \tag{15}$$

Proof. Let condition $(\star)$ be

$$\begin{aligned} &\sigma \in \mathcal{T}_P \text{ and } |\sigma| > n \text{ and} \\ &\qquad |\sigma| \geq i > 0 \text{ and } |(\psi\phi)^i(\sigma)| > n \text{ and } (\psi\phi)^i(\sigma)\upharpoonright n = \sigma\upharpoonright n. \end{aligned} \tag{16}$$

To prove the first claim it suffices to show that for all $n$ there exists $\sigma$ and $i$ satisfying $(\star)$. Fix $n$ and any $X \in P$. Observe that $(\psi\phi)^a$ is a tree map for all $a$. Let $A = \{(\Psi\Phi)^a(X) : a \in \omega\}$ and note $A \subseteq P$. If $A$ is finite then there exist $a$, $b$ with $a < b$ and $(\Psi\Phi)^a(X) = (\Psi\Phi)^b(X)$. If $A$ is infinite then, by compactness and the Pigeonhole Principle, there exist $a$, $b$ with $a < b$ and $(\Psi\Phi)^a(X)\upharpoonright n = (\Psi\Phi)^b(X)\upharpoonright n$. In either case let $s$ be large enough so that $|(\psi\phi)^a(X\upharpoonright s)| > n$ and $|(\psi\phi)^b(X\upharpoonright s)| > n$. Then $\sigma = (\psi\phi)^a(X\upharpoonright s)$ and $i = b - a$ satisfies $(\star)$.

If $Q$ is homogeneous, then, by Lemma 5.5, $\theta$ is a map from $\mathcal{T}_P$ to $\mathcal{T}_P$. As

$$\begin{aligned}\theta_{\phi,\psi}(n, f_{\phi,\psi}(n), \emptyset) = \psi(\phi(\emptyset) \,/\, \phi((\psi\phi)^{i-1}(f_{\phi,\psi}(n)))) = \\ \psi(\phi((\psi\phi)^{i-1}(f_{\phi,\psi}(n)))) = (\psi\phi)^i(f_{\phi,\psi}(n)),\end{aligned} \tag{17}$$

we find that $\theta_{\phi,\psi}(n, f_{\phi,\psi}(n), \emptyset)\restriction n = f_{\phi,\psi}(n)\restriction n$. ⊣

We can now diagonalize against pairs of partial computable functions $\langle\phi, \psi\rangle$ while ensuring that our witnesses share a cone above level $n$. To ensure that $n$ is a duplication level we copy that cone across level $n$.

THEOREM 7.5. *There exists a degree $\boldsymbol{a} \in \mathcal{L}_M$ which is hyperinseparable and not homogeneous.*

PROOF. We will construct a computable tree, $\mathbb{P}$, via a finite injury construction. The description begins with a listing of structures and related variables used in the construction. This is followed by a list of invariants which will be preserved at every stage. The class $P = [\mathbb{P}]$ will be such that $\deg(P)$ is hyperinseparable and not homogeneous. We identify the pair $\langle\phi, \psi\rangle$ with a code $e$ and, for simplicity, write $f_e$ and $\theta_e$ for $f_{\phi,\psi}$ and $\theta_{\phi,\psi}$ respectively.

We work to satisfy the following requirements:

$$\begin{aligned}\mathcal{R}_e\colon \exists n\Big[f_e(n)\downarrow \Rightarrow \big(\exists\tau \in \mathcal{T}_P\ \theta_e(n, f_e(n), \tau) \notin \mathcal{T}_P \\ \text{or } \exists X \in P\ \exists m\ \forall\sigma \prec X |\theta_e(n, f_e(n), \sigma)| < m\big)\Big],\end{aligned} \tag{18}$$

$$\mathcal{P}\colon \exists^\infty n\ [n \text{ is a duplication level for } P]. \tag{19}$$

Requirement $\mathcal{R}_e$ should understood to say that if $\theta_e$ appears to induce a Medvedev equivalence then it is with a non-homogeneous class, i.e., if $f_e(n)$ converges, and $\theta_e$ behaves like a functional then its range is not $P$.

Requirement $\mathcal{R}_e$ is of higher priority than $\mathcal{R}_d$ iff $e < d$.

The actual objects constructed are as follows.

- $\mathbb{P}$ is the tree defining the $\Pi^0_1$ class. Elements are enumerated into $\mathbb{P}$. $\mathbb{P}_s$ denotes the elements in $\mathbb{P}$ at the end of stage $s$.
- $l(s)$ is a total computable function defined such that all elements of length $l(s)$ or less have been added to $\mathbb{P}$ by the end of stage $s$.

The function $l(s)$ will ensure that $\mathbb{P}$ is a computable tree.

The following objects are used internally to construct the above.

- $L$ is the set of live strings. $L_s$ denotes the elements of $L$ at the end of stage $s$. At stage $s$, the only paths eligible for extension are those in $L_{s-1}$ of length $l(s-1)$. We will prove that $\mathrm{Ext}(\mathbb{P}) = L$.

In addition, for each requirement $\mathcal{R}_e$ we have the following.

- $n_e$ corresponds to the $n$ in the definition of $\mathcal{R}_e$, (18). $n_{e,s}$ denotes the value of $n_e$ at the end of stage $s$. In the absence of injury $n_e$ is constant once defined.
- $t_e$ is the "protection level" of requirement $\mathcal{R}_e$. It will also be a duplication level. $t_{e,s}$ denotes the value of $t_e$ at the end of stage $s$.

The following object is not required by the construction but will be tracked for use in the proofs of correctness.

- $\tau_e$ corresponds to the $\tau$ in the definition of $\mathcal{R}_e$, (18). $\tau_{e,s}$ will denote the value of $\tau_e$ at the end of stage $s$. In the absence of injury $\tau_e$ is constant once defined.

Finally, the following are not objects but rather subsets or notation for various values. These are not modified directly but rather reflect any changes of the objects they are based on.

- $L_s^l = \{\rho : \rho \in L_s \text{ and } |\rho| = l(s)\}$, i.e., the set of live leaves of $\mathbb{P}_s$.
- $L_s^{t_e} = \{\rho : \rho \in L_s \text{ and } |\rho| = t_{e,s}\}$, i.e., the set of live nodes at level $t_{e,s}$,
- $L^{t_e} = \lim_s L_s^{t_e}$.

The following invariants are maintained at all stages $s$ and are used in the proof of correctness. We need $\mathbb{P}$ to be computable and $L$ co-c.e. For every $s$,

$$l(s-1) \leq l(s), \tag{20}$$

$$\sigma \in \mathbb{P}_s \setminus \mathbb{P}_{s-1} \Rightarrow l(s-1) < |\sigma| \leq l(s), \tag{21}$$

$$\sigma \in L_s \setminus L_{s-1} \Rightarrow l(s-1) < |\sigma| \leq l(s). \tag{22}$$

We want $L$ to have no dead ends and describe live nodes. For every $s$,

$$\sigma \in L_s \text{ a leaf} \Rightarrow |\sigma| = l(s), \tag{23}$$

$$\sigma \in \mathbb{P}_s \setminus \mathbb{P}_{s-1} \Rightarrow \exists \rho \in L_{s-1}[\sigma \succ \rho]. \tag{24}$$

The interaction of requirements is controlled by the $t_e$'s. The $t_e$'s form an increasing sequence and will be the duplication levels. In addition, each $t_e$ defines the scope of protection of $\mathcal{R}_e$. Lower priority requirements are not allowed to kill any nodes below $t_e$. Our witnesses will be chosen above the previous protection level and below ours. For every $e$ and $s$,

$$d < e \text{ and } t_{d,s} \text{ defined and } t_{e,s} \text{ defined} \Rightarrow t_{d,s} < t_{e,s}, \tag{25}$$

$$\mathcal{R}_e \text{ not active at stage } s \Rightarrow L_s^{t_e} = L_{s-1}^{t_e}, \tag{26}$$

$$\forall \sigma \in L \forall \rho \in L_s^{t_e}[\sigma \,/\, \rho \in L], \tag{27}$$

$$t_{e,s} \geq n_{e,s} > t_{e-1,s} \text{ if } n_{e,s} \text{ defined}, \tag{28}$$

$$t_{e,s} \geq |\tau_{e,s}| > n_{e,s} \text{ if } \tau_{e,s} \text{ defined}. \tag{29}$$

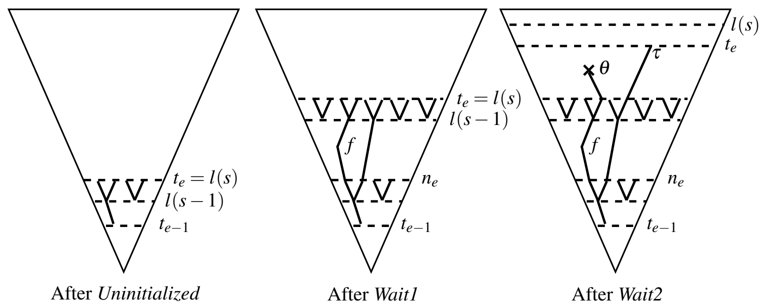


FIGURE 2. Strategy for $\mathcal{R}_e$

We treat all objects from Definition 7.3 as using $L$ for $\mathcal{T}_P$. This treatment is justified by our proof that $\mathcal{T}_P = \mathrm{Ext}(\mathbb{P}) = L$. For example, $f_e(n_{e,s})[s]$ uses $L_s$ for $\mathcal{T}_P$. Note that this guarantees that $f_e(n_{e,s})[s] \in L_s$.

Each strategy acts in three states, *uninitialized*, *wait1*, *wait2*, and then stops in the state *stop*. Whenever a strategy acts it will injure all lower priority strategies. In *uninitialized* the strategy splits all paths at the previous protection level and chooses an $n_e$. This technique of splitting is used repeatedly. The strategy will guarantee that at most one of the two splits are killed, thus preserving the common ancestors in $L$. In *wait1* the strategy waits for $f_e(n_e)$ to converge. If it does then the strategy ensures a split above $f_e(n_e)$ and raises the protection level to $|f_e(n_e)|$. In *wait2* the strategy waits for a $\tau$ to appear. If $\tau$ appears the strategy kills the corresponding $\theta$ and raises the protection level to $|\tau|$. See Figure 2 for a possible execution of a strategy.

At the beginning $\mathbb{P} = L = \emptyset$, all other variables are undefined, and all strategies are in the state *uninitialized*. At each stage $s$ the highest priority requirement $\mathcal{R}_e$ with $e \leq s$ that requires attention acts. For all other requirements $d$ that are not in the state *uninitialized*, $n_{d,s} = n_{d,s-1}$, $\tau_{d,s} = \tau_{d,s-1}$, and $t_{d,s} = t_{d,s-1}$. The strategy for $\mathcal{R}_e$ is injured when a higher priority strategy for $\mathcal{R}_d$ ($d < e$) changes $t_d$. When $\mathcal{R}_e$ is injured it is *reset*: it returns to the state *uninitialized* and $n_{d,s}$, $\tau_{d,s}$, and $t_{d,s}$ become undefined. The strategy is described by state below.

*Uninitialized*: $\mathcal{R}_e$ always requires attention. Our goal in this state is to choose $n_e$ and split all paths of $L_s^{t_{e-1}}$ to keep them alive. Let

$$L_s = L_{s-1} \cup \{\rho^\frown 0, \rho^\frown 1 : \rho \in L^l_{s-1}\}, \tag{30}$$

$$\mathbb{P}_s = \mathbb{P}_{s-1} \cup L_s, \tag{31}$$

$$t_{e,s} = n_{e,s} = l(s) = l(s-1) + 1. \tag{32}$$

In defining $t_e$ we reset all lower priority requirements. Note that we preserve all invariants. We enter the state *wait1*.

*Wait1*: $\mathcal{R}_e$ requires attention if $f_e(n_{e,s-1})[s-1]\downarrow$. We need to preserve $f_e$. To do so we need to raise our protection level and split $f_e$ so that we can preserve its life if it is later necessary to kill a child of it. We do this in a manner identical to the above. Let

$$L_s = L_{s-1} \cup \{\sigma^\frown 0, \sigma^\frown 1 : \sigma \in L^l_{s-1}\}, \tag{33}$$
$$\mathbb{P}_s = \mathbb{P}_{s-1} \cup L_s, \tag{34}$$
$$t_{e,s} = l(s) = l(s-1)+1, \tag{35}$$
$$n_{e,s} = n_{e,s-1}. \tag{36}$$

As before, the definition of $t_e$ causes all lower priority strategies to reset. Note that we preserve all invariants. We enter the state *wait2*.

*Wait2*: $\mathcal{R}_e$ requires attention if

$$\exists\tau \in L_{s-1}\big[|\tau| > n_{e,s-1} \text{ and} \tag{37}$$
$$\theta_e(n_{e,s-1}, f_e(n_{e,s-1})[s], \tau)[s-1] \in L_{s-1} \text{ and} \tag{38}$$
$$\tau\restriction t_{e-1,s-1} = f_e(n_{e,s-1})[s-1]\restriction t_{e-1,s-1} \text{ and} \tag{39}$$
$$\tau\restriction n_{e,s-1} \neq f_e(n_{e,s-1})[s-1]\restriction n_{e,s-1} \text{ and} \tag{40}$$
$$|\theta_e(n_{e,s-1}, f_e(n_{e,s-1})[s-1], \tau)[s-1]| > t_{e,s-1}\big]. \tag{41}$$

We want $\tau$ to witness $\mathcal{R}_e$ [(37) and (38)]. Also, $\tau$ should be in the same cone above $t_{e-1}$ [(39)] but in a different subcone than $\theta_e(n_e, f_e(n_e), \tau)$ [(40)] so that $\tau$ can stay alive when we kill $\theta_e(n_e, f_e(n_e), \tau) \succeq f_e(n_e)$. Finally, we want $\theta_e(n_e, f_e(n_e), \tau)$ to be above our split over $f_e(n_e)$ so that killing it does not kill $f_e(n_e)$ [(41)]. If such a $\tau$ exists then we kill the resulting $\theta_e(n_e, f_e(n_e), \tau)$ and duplicate the kill across all members of $L^{t_{e-1}}_{s-1}$. To ensure that $L_s$ has no dead ends we need to remove a several strings from it, namely, to kill a path $\sigma$ we need to remove $\sigma$ and all ancestors of $\sigma$ up to the nearest split. Define the *kill set* of $\sigma$ by

$$ks(\sigma) = \{\epsilon \in L_{s-1} : \forall\delta \in L_{s-1}[\delta \succeq \epsilon \Rightarrow \delta \preceq \sigma]\}.$$

For simplicity of notation let $\sigma = \theta_e\big(n_{e,s-1}, f_e(n_{e,s-1})[s], \tau\big)[s]$. Let

$$\tau_{e,s} = \tau, \tag{42}$$
$$L_s = L_{s-1} \setminus \bigcup_{\gamma \in L^{t_{e-1}}_{s-1}} ks(\sigma \,/\, \gamma), \tag{43}$$
$$\mathbb{P}_s = \mathbb{P}_{s-1}, \tag{44}$$
$$l(s) = l(s-1), \tag{45}$$
$$t_{e,s} = \max\{t_{e,s-1}, |\tau|\}, \tag{46}$$
$$n_{e,s} = n_{e,s-1}. \tag{47}$$

We reset all lower priority requirements and enter state *stop*. Observe that all invariants are preserved.

*Stop*: $\mathcal{R}_e$ never requires attention.

CLAIM. *Every requirement acts a finite number of times.*

PROOF. In the absence of injury each requirement acts at most three times, once each in the states *uninitialized*, *wait1*, and *wait2*. Thus, in typical finite injury fashion, every requirement acts a finite number of times. ⊣

CLAIM. *For all $s$, $L_s \neq \emptyset$.*

PROOF. At stage 0, $\mathcal{R}_0$ is in state *uninitialized* and requires attention. It sets $L_0 = \{0, 1\}$ and $t_{0,0} = 1$. As there are no higher priority strategies $\mathcal{R}_0$ will never be injured and thus $t_{0,0}$ will never be undefined. At all later stages 0 and 1 will be protected by $t_{0,s}$ and thus $\{0, 1\} \subseteq L_s$. ⊣

CLAIM. $\lim_s l(s) = \infty$ *and* $\lim_s L_s = \text{Ext}(\mathbb{P})$.

PROOF. By the previous Lemma, for all $s$, $T_s \supseteq L_s \neq \emptyset$. As each requirement acts a finite number of times every requirement acts at least once in the state *uninitialized*. During this action $l(s)$ is increased by 1. Thus $\lim_s l(s) = \infty$. As paths leave $L$ at most once, $\lim_s L_s$ is well defined. If $\tau \in L_s$ for all $s$, then as $L_s$ has no leaves shorter than $l(s)$, for any $n$ we can find $s$ such that $l(s) > n$ and thus $\tau$ has a child of length $n$. As any elements added to $L$ are added to $\mathbb{P}$ we have $\tau \in \text{Ext}(\mathbb{P})$. Conversely, if $\tau \in \text{Ext}(\mathbb{P})$ then it must have children of all lengths. Let $s$ be such that $\tau \in L_s \setminus L_{s-1}$. If $\tau$ ever left $L$ after stage $s$ then by the invariants it could have no more children enter $\mathbb{P}$. But $\tau$ has infinitely many children, a contradiction. Thus $\text{Ext}(\mathbb{P}) \subseteq \lim_s L_s$. So $\text{Ext}(\mathbb{P}) = \lim_s L_s$. ⊣

CLAIM. *Every requirement $\mathcal{R}_e$ is satisfied.*

PROOF. Assume otherwise. As $\mathcal{R}_e$ acts a finite number of times, let $s$ be such that $\mathcal{R}_e$ acts last at stage $s$. We now have three cases, depending on which state $\mathcal{R}_e$ acted in at stage $s$.

Case 1: If $\mathcal{R}_e$ acted in state *uninitialized* then $f_e(n_{e,s-1}) \uparrow$, otherwise there would be some $t$ such that $f_e(n_{e,s-1})[t] \downarrow$ and $\mathcal{R}_e$ would have acted later. So $n_e$ is defined and $f_e(n_e) \uparrow$ so $\mathcal{R}_e$ is satisfied.

Case 2: If $\mathcal{R}_e$ acted in state *wait1* then $f_e(n_{e,s})[s] \downarrow$. Observe that $n_e = n_{e,s}$, $f_e(n_e) = f_e(n_{e,s})[s]$, and, by definition, $|f_e(n_e)| > n_{e,s}$ and thus, by invariants, $|f_e(n_e)| > t_{e-1,s} = t_{e-1}$. Let $\gamma = f_e(n_e) \upharpoonright t_{e-1}$. Let $s'$ be the last stage that $\mathcal{R}_e$ acted in state *uninitialized*, and note that in state we put $\gamma^\frown 0$ and $\gamma^\frown 1$ into $L_{s'}$ and raised the protection level, $t_{e,s'}$, to $|\gamma^\frown 0|$. As $\mathcal{R}_e$ was not injured after stage $s'$ both $\gamma^\frown 0$ and $\gamma^\frown 1$ are in $L$ and, as a result, have children of arbitrary length. Without loss of generality we can assume that $f_e(n_e) \succeq \gamma^\frown 0$. Fix $X \in P \succeq \gamma^\frown 1$, such exists as $f_e(n_e) \in L^{t_e}$. If $|\theta_e(n_e, f_e(n - e), \sigma)| < m$ for some $m$ and all $\sigma \in P$ then $\mathcal{R}_e$ is satisfied. Assume otherwise and assume that $\mathcal{R}_e$ is not satisfied. Then $\forall \tau \in \mathcal{T}_P[\theta_e(n_e, f_e(n_e), \tau) \in \mathcal{T}_P]$. There is some child $\tau$ of $\gamma^\frown 1$ which is long enough to satisfy the attention requirements of state *wait2*. But then $\mathcal{R}_e$ would have acted in *wait2*, a contradiction. Thus $\mathcal{R}_e$ must be satisfied.

Case 3: If $\mathcal{R}_e$ acted in state *wait2*, then it has killed $\tau_e$. Thus $n_e$ and $\tau_e$ witness the satisfaction of $\mathcal{R}_e$. All we need to do is show that $f_e(n_e)$ and $\tau_e$ are alive. By the invariants, no requirements of lower priority can kill them, and as $\mathcal{R}_e$ is uninjured after stage $s$ no higher priority requirements killed $f_e(n_{e,s})$ or $\tau_{e,s}$. Both are in the same cone above $\gamma \in L^{t_{e-1}}$ so we only need to worry that the killing of $\gamma = \theta_e(n_e, f_e(n_e), \tau_e)$ killed either. Killing $\gamma$ cannot kill $\tau_e$, as by the conditions of state *wait2* they are incomparable. Those conditions also require $\sigma$ to be above the split above $f_e(n_e)$ so killing $\gamma$ will kill at most one of $f_e(n_e)^\frown 0$ and $f_e(n_e)^\frown 1$. Thus all witnesses are preserved and $\mathcal{R}_e$ is satisfied. ⊣

Claim. *The requirement $\mathcal{P}$ is satisfied.*

Proof. For any $n$, by the invariants, there exists $e$ such that $t_e > n$. Fix such an $e$ and let $s$ be large enough that $\mathcal{R}_e$ never acts after stage $s$. Then $t_{e,s}$ is a duplication level as $L^{t_e}$ will not change and all actions after stage $s$ will preserve the invariant that $L^{t_e}$ is a duplication level. ⊣

So $P = [\mathbb{P}]$ satisfies the requirements. By Lemma 7.2, $P$ is hyperinseparable. If $P$ is in a homogeneous degree then there exist $\phi, \psi$ as in Lemma 7.4. But then $\mathcal{R}_{\phi,\psi}$ guarantees that $n_{f,g}$ and $\tau_{f,g}$ witness a contradiction. Thus $P$ is not a homogeneous degree so $\deg(P)$ is a hyperinseparable and not homogeneous degree. ⊣

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